PAPER III

Gr A : DISCRETE MATHEMATICAL STRUCTURES

1. Define *graph, simple graph, finite graph, null graph, pendant vertex, complete* graph, *connected graph, connected component, walk, path, trail, cycle, Hamiltonian cycle, Eulerian closed trail, tree, degree, eccentricity, centre, radius(r), diameter(d), girth (length of the smallest cycle), cutset/cocycle, weight, centroid, spanning tree, chord, bipartite graph, planar graph, plane graph and clique.*
2. Define *directed graph, semi-cycle, semi-path, semi-walk, strongly connected digraph, weakly connected digraph, disconnected digraph directed tree, strong component and weak component, Euler dirgraph and tree in a digraph.*
3. Begin with a filled up vessel of 8 liter (say, A) and two empty vessels of 5 liter (B) and 3 liter (C). Divide the liquid in A into two equal quantities.
4. Draw a graph with 64 vertices to represent the squares of a chessboard. Join two vertices representing the move of a knight. Determine the number of vertices with degree two, three, four, six or eight.
5. Euler's theorem for a planar graph states that a graph with n vertices and e edges has e-n+2 faces or regions. Prove this theorem. A soccer ball is constructed out of regular pentagons and regular hexagons of equal edges. If each arm of a pentagon is a common arm of a hexagon and each alternate arm of a hexagon is a common arm with a pentagon, find the geometry of the soccer ball. (*Remark:* Same geometry applies to buckminister- fullerene, a super-conducting polymorphism of carbon)
6. Now divide the soccer-ball into two hemispheres (without affecting any pentagon/hexagon). Reconnect the halves at the two ends of a cylindrical shell (made out of 6 hexagons) to get a Rugby ball (Its not spherical rather a spheroid). Determine the number of vertices in a Rugby ball. If you go on extending the cylindrical shell you will get the geometry of a *nanotube*, a cylindrical molecules of fullerne.
7. How many graphs can be constructed with n vertices?

8. How many binary trees can be constructed with n vertices?

9. Determine the *Euler digraph* for a *k-cube*. A *k-cube* is a graph whose vertices are ordered k-tuples of *0*s and *1*s. Two vertices are connected iff they differ exactly in one coordinate. Show that a k-cube is bipartite. Also determine the vertex set of each partition.

10. Define *arborescence, oriented tree and binary tree (topological bifurcating arborescence).*

11. Prove the following theorems:

i) the sum of the degrees of all the vertices in an undirected graph is twice the number of edges.

ii) the number of vertices of odd degree in a graph is always even.

iii) Maximum number of edges in a simple graph with n vertices is $n(n-1)/2$.

iv) If a graph has exactly two vertices of odd degree, then there must be a path joining these two vertices.

v) A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

vi) If G be a graph with n vertices, then the following statements are equivalent.

a) G is a tree.
b) Any two vertices in G s connected by a unique path.
c) G is connected and the number of edges in G is $n-1$.
d) G is *acyclic* and the number of edges in G is n-1.
e) G is a minimally connected graph.
f) G has no cycle but connecting any two vertices with an edge will create a cycle.

vii) Let G be a tree with n_i number of vertices with degree i, where $0 \le i \le m$, m being the maximum degree of a vertex. Prove that total number of leaf nodes will be

$$n_0 = 1 + \sum_{k=2}^{m} (k-1)n_k.$$

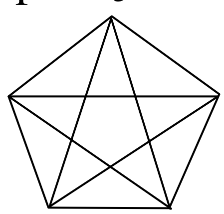
viii) Every tree contains at least two vertices of degree 1 $(n>1)$.

ix) A tree can have at most two centres.

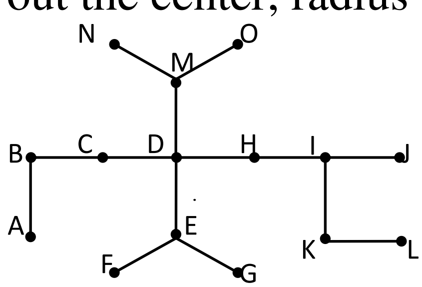
x) In an extended binary tree with n_e external nodes (i th external node has a level l_i)

$$\sum_{i=1}^{n_e+1} \mathbf{2}^{-l_i} = \mathbf{1}$$

12. Prove that either **G** or complement of **G** will be connected.
13. Show that deletion of an edge can increase the number of components by at most unity.
14. Show that there is a *u*-*v* walk in **G**, iff there is a *u*-*v* path.
15. A *k- regular* graph of *girth* (length of the smallest cycle) *five* has k^2+1 vertices or more.
16. Each vertex of a connected graph **G** has an even degree. For any $v \varepsilon$ **V** show that $w(\mathbf{G}-v) \leq \frac{1}{2}\, degree(v)$
14. If the minimum degree (δ) of a graph **G** is two or more, then show that **G** contains a cycle of length $\delta+1$ or more.
15. Find out Euler trail and a Hamiltonian cycle for the following *Kuratowski* graph $\mathbf{K_5}$.



16. How many vertices are there in the *Kuratowski* graphs $\mathbf{K_n}$ and $\mathbf{K_{m,n}}$?
17. Describe the complement graph of $\mathbf{K_{m,n}}$.
18. Find out the center, radius and diameter for the following graph.



19. Define *incidence-matrix*, *adjacency-matrix* and *adjacency-list*.

20. Show that $r \leq d \leq 2r$, r and d are the *radius* and *diameter* of the graph.

21. Show that $\delta \leq 2e/v \leq \Delta$

where δ is the *min-degree*

Δ is the *max-degree*

e is the number of edges.

22. Show that the $g \leq 2d+1$, where g is the girth of Graph **G** and d is the *diameter.*

23. Prove that a graph is bipartite if it does not have any *odd cycle*.

24. Prove that the diagonal elements of $\mathbf{A}^n$ of a bipartite graph **G** are all zero.

25. If a *k- regular* graph has bipartitions **X** and **Y** then $|\mathbf{X}| = |\mathbf{Y}|$

26. If a simple and connected graph **G** has no induced *subgraph* containing *two* edges then prove that **G** is complete.

27. From graph theoretic approach show that

$$^{n}\mathbf{C}_2 = {}^{k}\mathbf{C}_2 + k\,(n-k) + {}^{n-k}\mathbf{C}_2$$

28. Show that every tree with exactly two out-degree vertices is a path.

29. Show that if **G** is a tree with $\Delta = k \geq 2$ then **G** has at least k vertices of degree **1.**

30. If T be an arbitrary tree of n vertices, and **G** be a graph with $\delta \geq n-1$. Then prove that **G** contains a subgraph isomorphic to T.

31. Show that you can give an enumeration of the vertices of a tree **T**.

32. Prove that an edge e of **G** is a cut edge iff e is contained in no cycle.

33. Addition of any edge e to a given tree **T**, *i.e.* T+e contains a unique cycle.

34. Show that a graph is bi-connected iff every two vertices of that graph be connected with two internally disjoint paths.

35. A connected graph is Eulerian if and only if it has atmost two vertices of odd degree.

36. Prove that a matching **M** is maximum iff **G** contains no ***M**-augmented path.*

37. How many maximum matching can be there in $\mathbf{K_n}$

38. If **M** and $\mathbf{M}^{/}$ be two matchings of a graph **G**, then prove that each component of

$\mathbf{M \Delta M}^{/}$ is either a *path* or a *cycle*.

39. Show that a tree has at most one perfect matching.

40. State and prove the Hall's theorem.

41. If **G** is a *k- regular bipartite* graph with k > 0, then prove that **G** has a perfect matching.

42. A set $\mathbf{S \subseteq V}$ is an independent set iff $\mathbf{V \backslash S}$ is a vertex cover of **G**.

43. Write down the following graph algorithms and illustrate with appropriate example. Find out the complexity of each algorithm.

i) Prim's algorithm to determine Minimum Spanning Tree of a graph.

ii) Dijkstra's algorithm for finding out the shortest paths from a vertex to all other vertices.

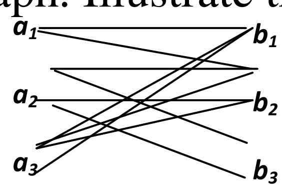
iii) Floyd's algorithm to finding out shortest paths in between all pair of vertices.

iv) Warshall's algorithm for finding out the reachability of a vertex from all other vertices.

v) Breadth-first-search algorithm for a graph

vi) Depth-first-search algorithm for a graph.

vii) Hungarian algorithm to determine the maximal matching of a bipartite graph. Illustrate the same with the following graph.



44. Illustrate DFS and BFS on a binary tree.

45. Use BFS and DFS to test reachability (and path) and connectedness of a graph.

46. Prove the De Morgan's laws in set theory, viz

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

47. What do you mean by *power-set* of a set? Prove by mathematical induction that if a set has n elements, then its power-set has 2^n elements.

48. A pizza shop serves pizzas with any combination of the following ingredients:

Mushrooms, pepperoni, peppers, sardines, sausage, anchovies, salami, bacon and peppers.

The shop claim t ffer over 500 verities. The consumer protection bureau does not agree. Who is right?

49. In a room of 100 people, 60 can speak Bengali, 30 can speak English, 40 can speak Hindi, 10 can speak both Bengali and English, 15 can speak both English and 20 Hindi, 20 can speak both Bengali and Hindi and 5 can speak all the three languages. How many of them can speak none of the languages?

50. When does a function have an inverse function?

51. Let $g : A \to \mathbf{B}$ be a function and let

a) $|\mathbf{B}| = |\mathbf{A}|$, b) $|\mathbf{B}| < |\mathbf{A}|$, c) $|\mathbf{B}| > |\mathbf{A}|$

a. In which cases does g has an inverse and why?

b. Verify whether the following functions are *one-to-one (injective)* and *onto (surjective)* where R is the *set of real numbers* and Q is the *set of rational numbers.*

i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x$

ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$

iii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin^2 x + \cos^2 x$

iv) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$

v) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 \text{ if } x \geq 0$

$= -1 \text{ if } x < 0$

vi) $f : \mathbf{Q} \to \mathbf{Q}$ defined by $f(x) = 5x + 3$

52. When is a function called bijective? Give an example.

53. What is pigeonhole principle?

Use it to prove that a function f : $\mathbf{A} \to \mathbf{B}$ cannot be one-to-one if $|\mathbf{A}|$ is greater than $|\mathbf{B}|$.

54. Find out whether the following relations are equivalent relations are not?

i) $x \rho y$ if and only if $x = y$, $\forall x,y \in \boldsymbol{R}$

ii) $x \rho y$ if and only if $x \leq y$, $\forall x,y \in \boldsymbol{R}$

iii) $x \rho y$ if and only if $x - y$ is divisible by 5, $\forall x, y \in \boldsymbol{I}$

(R is the set of real numbers and I is the set of integers)

55. Find out the equivalence classes for the equivalence relations above.

56. Prove the following by mathematical induction or otherwise.

a) $S(n) = 1 + 2 + 3 + \ldots + n = n(n+1)/2$

b) $5^n - 4n - 1$ is exactly divisible by 4, n is a non-negative integer.

c) $n^3 + 2n$ is exactly divisible by 3, n is a non-negative integer.

d) $3^n > 2n$, n is a non-negative integer.

57. Let A be a set with 8 elements. How many three-element subsets does A have?

58. Let X = { a, b, c }. Count the number of ordered samples of size four from X, where a) Repetition is allowed, b) Repetition is not allowed

59. Consider the word ELEVEN. Find out the following.

i) The total number of permutations of the word.

ii) How many of them begin and end with E?

iii) How many of them have the three E's together?

iv) How many of them begin with E and end with N?

60. Let $f : \mathbf{A} \rightarrow \mathbf{B}$ be a function and let $|\mathbf{A}| = m$ and $|\mathbf{B}| = n$. Then how many such functions f are possible? How many of them are i) one-to-one and ii) onto?

61. How many four-digit decimal numbers are there? How many of them have no digit lower than 3?

62. How many six-digit decimal numbers with no leading 0 are there which do not repeat a digit? How many of them are divisible by 5?

63. A number like 23132, which reads the same forward and backward, is called a palindrome. Answer the following.

64. How many three-digit palindromes are there in radix 10?

65. How many n-digit palindromes are there in radix 10?

66. How many n-digit palindromes with no leading 0's are there in radix 10?

67. A number consisting of the bits 0 and 1 only is called a binary number.

68. How many n-bit binary numbers are there?

69. How many n-bit palindromes are there in radix 2?

70. How many n-bit palindromes with no leading 0's are there in radix 2?

71. How many ternary numbers can be represented by the sequence a_0 $a_1\, a_2 \ldots a_{n-1}$,

72. $a_i \in \{0,1,2\}$ for $i=0,1,2, \ldots n-1$. If the leftmost position is reserved for sign, then how many positive *ternary* numbers can be represented by the sequence?

73. A box of n distinct marbles are marked *1.. n*. How many distinct sequence of k marbles can be chosen

   i) if the chosen is returned back to the box.

   ii) if the chosen is not returned back to the box.

74. A box of n distinct marbles are marked *1.. n*. How many unordered sequence of k marbles can be chosen i) if the chosen is returned back to the box.

   ii) if the chosen is not returned back to the box

75. In how many ways k marbles can be chosen from a box containing m identical *red* marbles and n identical *blue* marbles.

76. State and prove the generalized principle of inclusion and exclusion. An unmindful office bearer is carrying n letters for n different addressees. How many ways all the letters can be dispatched to all the addresses, with none of them getting the right one?

77. The prime numbers less than or equal to 10 are 2,3,5 and 7. Determine the number of primes lying in between 1 and 100.

78. Find the generating function for the infinite sequence 1, λ, λ^2, λ^3,…, where λ is a fixed constant.

79. Solve the following recurrence relations.

   i) $y_n + 2y_{n-1} - 15y_{n-2} = 0$ for $n \geq 2$ and $y_0 = 0$, $y_1 = 1$.

ii) $y_n - y_{n-1} - y_{n-2} = 0$ for $n \geq 2$ and $y_0 = 0$, $y_1 = 1$.

iii) $2x_{n+1} - x_n = (1/2)^n$, for $n \geq 1$ and $x_1 = 2$.

80. How many distinct integral solutions are there for

$$x_1 + x_2 + x_3 + x_4 = 20$$

subject to restrictions $0 \leq x_1$, $x_2 \leq 5$; $2 \leq x_3 \leq 6$; $x_4 \geq 13$.

81. Solve the Inhomogeneous Difference Equation (IDE):

$$2x_{n+1} - x_n = (1/2)^n, \text{ given that } x_1 = 2.$$

82. How many ways can a moneychanger give changes to a hundred-rupee note into two-rupee notes, five-rupee notes, ten-rupee notes, twenty-rupee notes and fifty-rupee notes?

83. Count the number of labeled graphs with $n \geq 2$ vertices and $e \leq {}^nC_2$ edges.

84. Use generating function concept to determine the number of unordered samples of size k from a box of n distinct articles.

85. What do you mean by worst-case time complexity of an algorithm?

86. Define Big–Oh(O), Big–Omega(Ω) and Big–Theta(θ) notation. Give examples.

87. Prove that $g(n)=\Omega(f(n))$ if and only if $f(n)=O(g(n))$.

88. Distinguish in between time-complexity and space-complexity of an algorithm. What is the minimum space-time requirement for adding up n numbers.

89. There are two algorithms-one is exponential-time and another is polynomial-time, which one is faster and why?

90. Out of the two approaches to generate a sequence of Fibonacci numbers iterative and recursive, which one would you prefer? Give reasons.

91. Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

92. Define Power Set?

93. What do you mean by Injective Mapping? Give example.

94. Define Mutually Exclusive and Exhaustive Events.

95. What is noise level?

96. Define Classical Definition of Probability. What are its limitations?

97. An article manufactured by a company consists of two parts I and II. In the process of

manufacture of part I, 9 out of 100 are likely to be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part II. Calculate the probability that the assembled article will not be defective.

98. State & prove the Distributive Property (Set Theory) (not by Venn Diagram).

99. State & prove the De Morgan's Laws (Set Theory) (not by Venn Diagram).

100. Define Cartesian Product of a Set with an example?

101.What do you mean by Equivalence Relation?

102. A Relation **ρ** is defined on the set Z (Set of all Integers) by

"***aρb if and only if a-b is divisible by 5*** "***for a, b ∈ Z***

Examine if **ρ** is (i) Reflexive (ii) Symmetric (iii) Transitive

103. Let S={ 1,2,3,4} , T={a,b,c,d}

Explain with reason whether the following relations f_1 and f_2 between S and T are Mappings from S to T.

(i) f_1={(1, a),(1, b),(2, c),(3, c),(4, d)}
(ii) f_2= {(1, b), (2, b), (3, c), (4, d)}

104. Define Mutually Exclusive and Exhaustive Events.

105. Give a big-O estimate for $f(x)=(x+1)\log(x^2+1)+3x^2$?

106. Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.Use the principle of Inclusion and Exclusion

107. State & Prove the Generalized Pigeon-Hole Principle

108. Which of the following are valid arguments? Justify your answer:

(i)(P→Q), (~Q→R), ~R, ∴P

(ii)(A → (B→C)), B, ∴ (A→C)

109. What are Existential and Universal Quantifiers? Illustrate with suitable example

110. What is Proposition? What are the five basic Connectives?

111. What is Tautology and Absurdity?

112. Prove that i) $[(p \wedge \sim q) \rightarrow r] \rightarrow [p \rightarrow (q \vee r)]$ is a Tautology.

ii) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$

113. i) Is the following a Probability Density Function (P.D.F)?

$$f(x) = \begin{cases} 2x, & 0<x\leq 1 \\ 4-2x, & 1<x\leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

ii) If the Random Variable X has the P.D.F

$$f(x) = \begin{cases} \frac{1}{4}, & -2\leq x\leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

obtain P {(2x+3)>5}. (Here P denotes Probability)

114. If $p \rightarrow q$ is a proposition, then construct the truth table for its Opposite, Converse and Contrapositive expression

Gr B: Numerical Methods and Algorithms

1. 'Gaussian elimination method can be used to solve any set of linear equations.' – Justify.

2. What is partial pivoting? Write an algorithm to solve the system of linear equations **AX** = **B**, [*Hints*: at first reduce the augmented matrix [**A**, **B**] to upper-triangular form and then perform back-substitution to get the final solution.]

3. Use *Gaussian* elimination *Method* to determine the inverse of **A**. Verify your answer.

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 1 & -5 & 3 \end{bmatrix}$$

4. Solve the system of linear equations by *iteration method.*

$$3x - y = 5$$

$$x + 2y = 4$$

5. Solve the equation $x^2 - \sin x = 5$ by *Newton-Raphson method.*

6. What are the norms that can be used to measure the distance of the curve $y = f(x)$ from the given data-set, in *least-square curve-fitting?*

7. Find the curve $y = Cx^a$ which passes through the points (1, 0.6), (2, 1.9), (3, 4.3), (5, 12.6). Use the change of variables $X = \ln(x)$, $Y = \ln(y)$ to linearise the data points.

8. Use *Runge-Kutta method* to solve the equation $dy/dx = -xy$, at $y(x=0.2)$ given $y(x = 0)=1$.

9. Compare and contrast between *Runge-Kutta method* and *Euler method* for solving differential equations.

10. What is noise level?

11. In an examination the number of candidates who secured marks between certain limits were as follows:

Marks :	0-19	20-39	40-59	60-89	80-99
No. of candidates :	41	62	65	50	17

Estimate the number of candidates getting marks less than 70.

12. Find the number of significant figures in V_A=1.8921, given its relative error as $0.1x10^{-2}$; V_A being the approximate value.

13. What do you mean by 'Round off Errors' & 'Truncation Errors' in Numerical Data?

14. Use Lagrange's Interpolation to find the value of ƒ (x) for x=2; given the following table:

x:	0	1	3	4
ƒ (x):	5	6	50	105

15. Evaluate $\int_0^1 (4x-3x^2)dx$, taking 10 intervals, by Trapezoidal Rule. Find the absolute & relative errors in your result.

16. Use Lagrange's Interpolation to find the value of ƒ(x) for x=0; given the following table:

x:	-1	-2	2	4

ƒ(x):	-1	-9	11	69

17. Evaluate $\int_{1.2}^{1.6} (x+1/x)\, dx$, correct upto 2 significant figures, taking 4 intervals, by

(i)Trapezoidal Rule, (ii) Simpson's one-third Rule

18. What is Interpolation? Derive Newton's Forward Interpolation Formula

(Without the error term)

20. Find, by suitable interpolation formula, the value of ƒ (1.42) from the

following table (correct up to three significant figures)

x:	1.1	1.2	1.3	1.4
ƒ (x):	7.831	8.728	9.697	10.744

21. Write down the approx. value of π / 4 correct up to four significant

figures and then find Relative Error.

22. What is Bisection method? Discuss Bisection method for finding a simple real root of an equation ƒ (x) =0 lying in the interval [a, b].Show that the method is certain to coverage.

24.Write the algorithm for Newton Raphson method

25. Use Lagrange's Interpolation to find the value of ƒ (x) for x=102; given the

following table:

x:	93.0	96.2	100.0	104.2	108.7
f (x):	11.38	12.80	14.70	17.07	19.91

26. What do you mean by System of non-homogeneous Linear Equations?
27. What do you mean by Noise Level?
28. Compute the values of the unknowns in the system of equations by Gauss-Elimination Method: (correct up to 4 significant figures)

$$x_1+2x_2+3x_3=10$$

$$x_1+3x_2-2x_3=7$$

$$2x_1-x_2+x_3=5$$